Digital Communication Systems

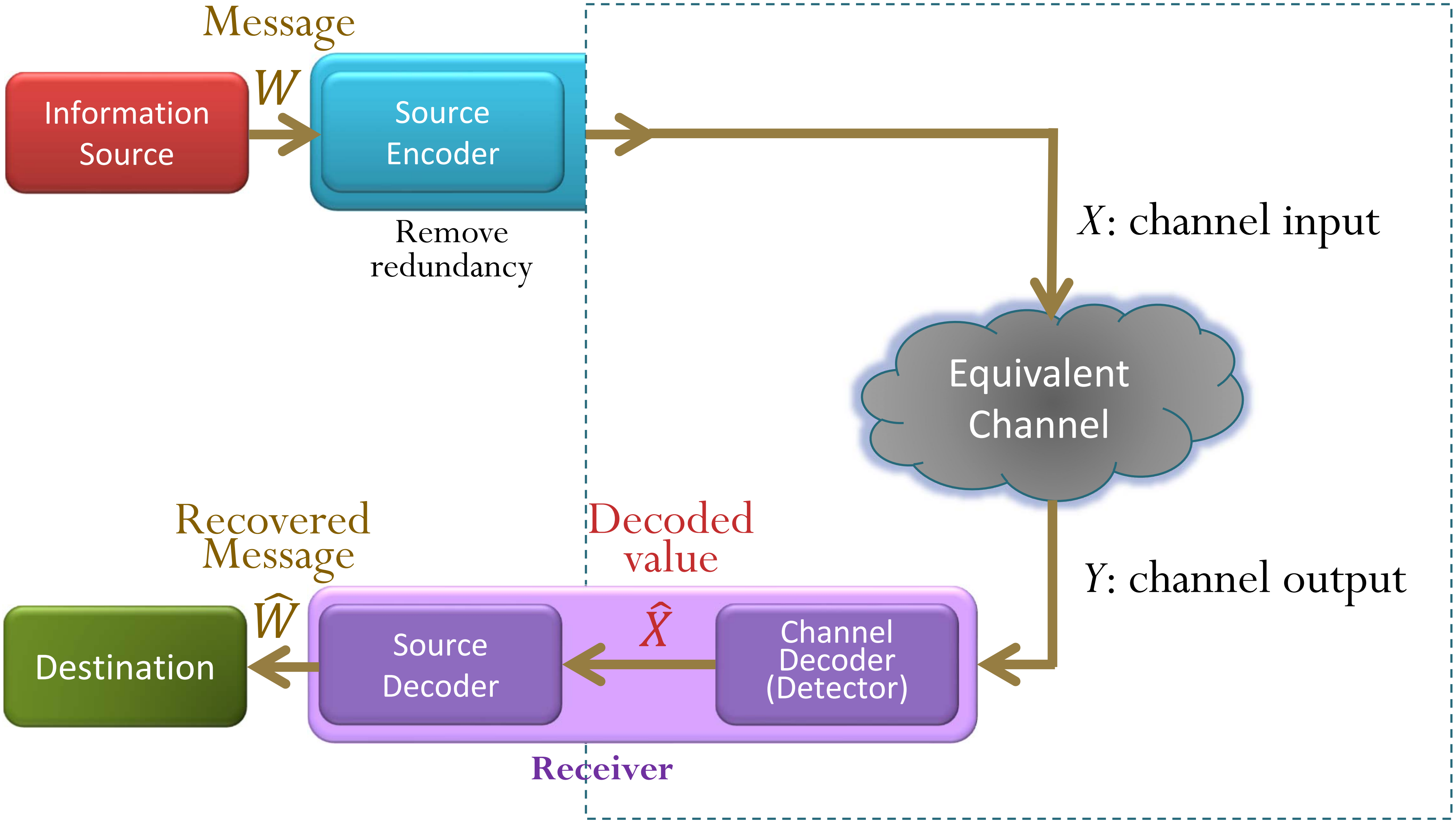
EES 452

Asst. Prof. Dr. Prapun Suksompong
prapun@siit.tu.ac.th

3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.5 Introduction to Channel Coding in Communications Over BSC

System Model for Section 3.2-3.3



Message

W

Information Source

Source Encoder

Remove redundancy

X: channel input

Equivalent Channel

Y: channel output

Channel Decoder (Detector)

Decoded value

$\hat{X}$

Source Decoder

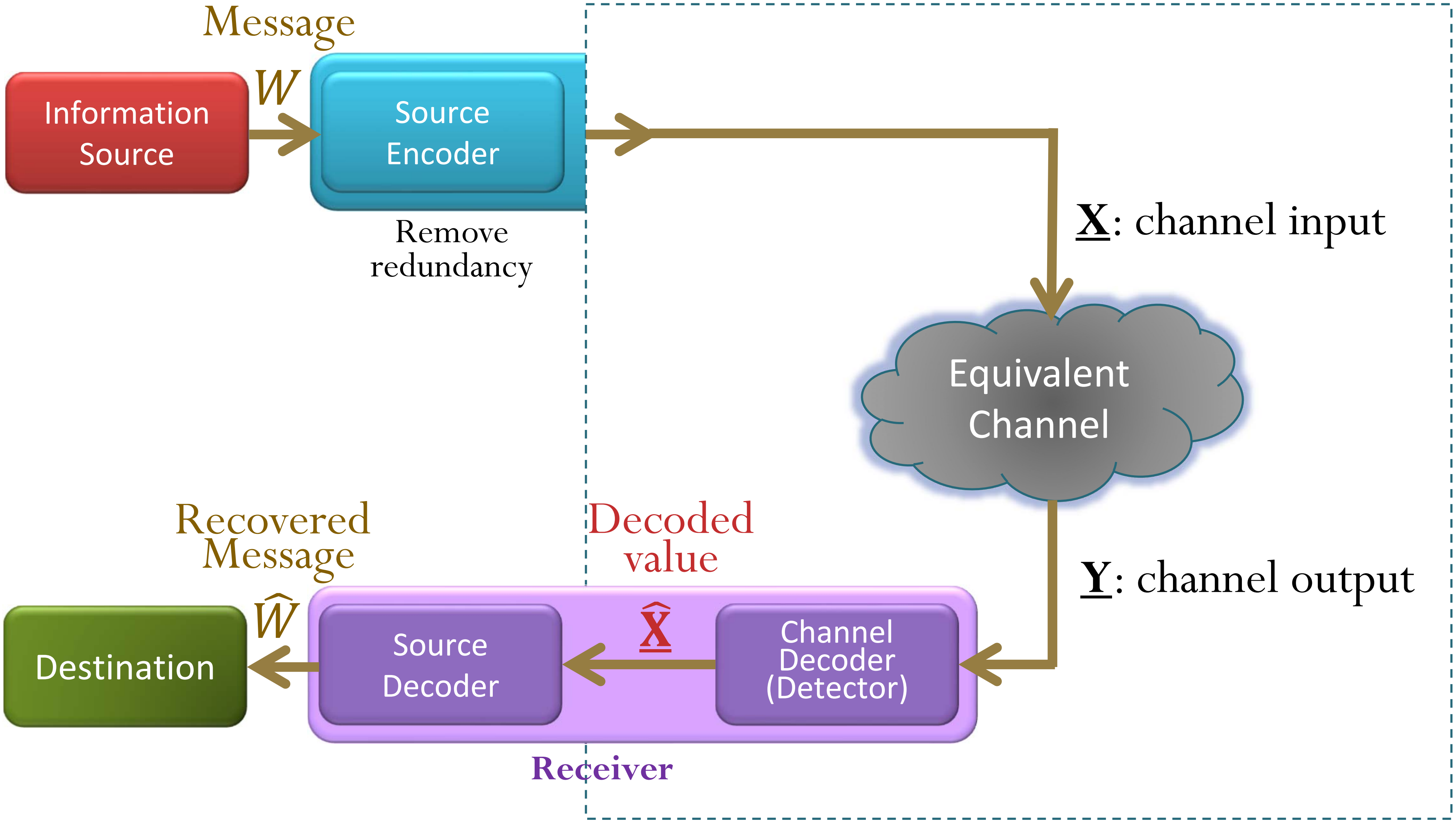
Recovered Message

$\widehat{W}$

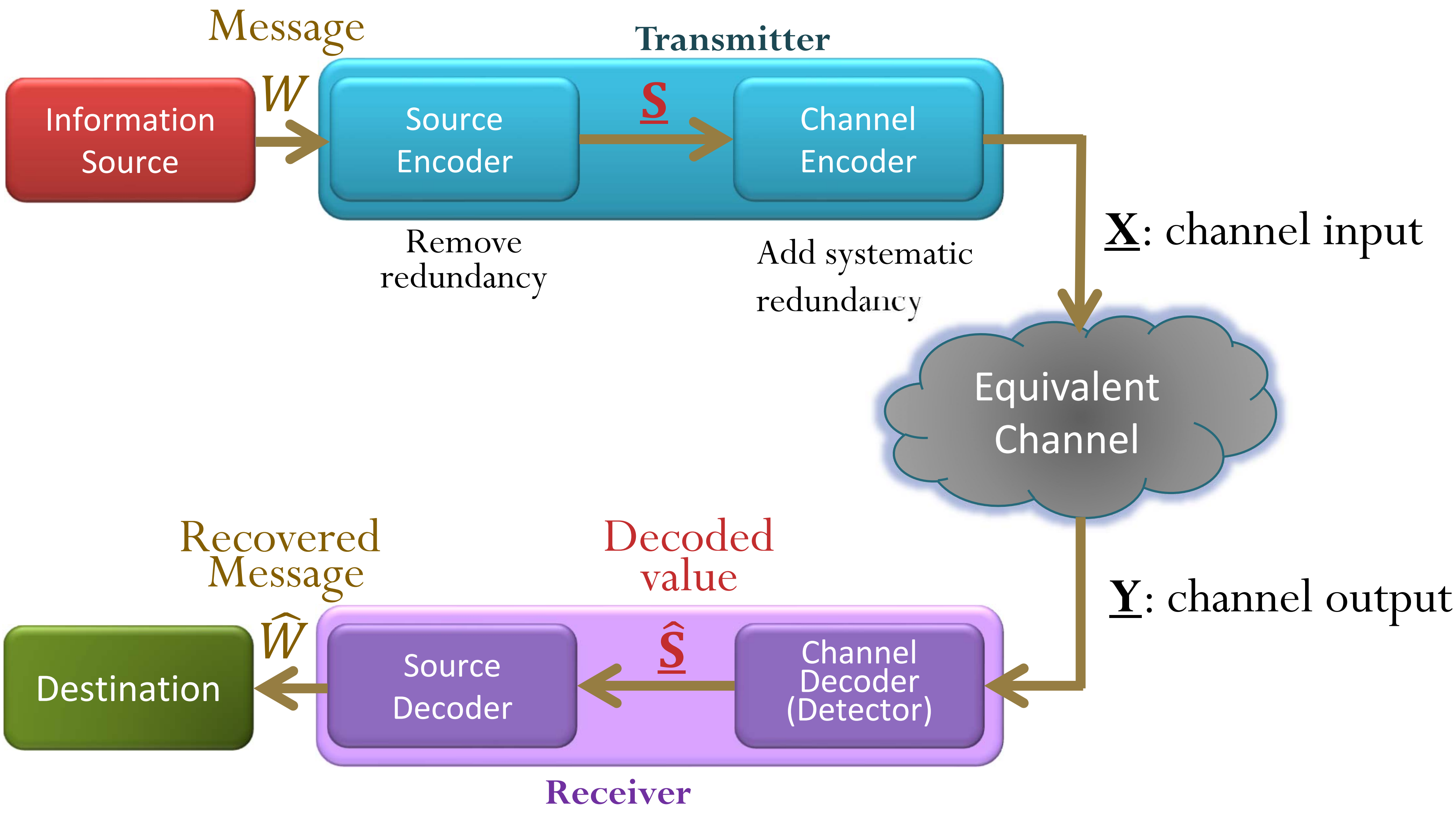
Destination

Receiver

System Model for Section 3.4



System Model for Section 3.5



Message

W

Information Source

Transmitter

Source Encoder

$\underline{\mathbf{S}}$

Channel Encoder

Remove redundancy

Add systematic redundancy

$\underline{\mathbf{X}}$: channel input

Equivalent Channel

$\underline{\mathbf{Y}}$: channel output

Recovered Message

$\hat{W}$

Destination

Source Decoder

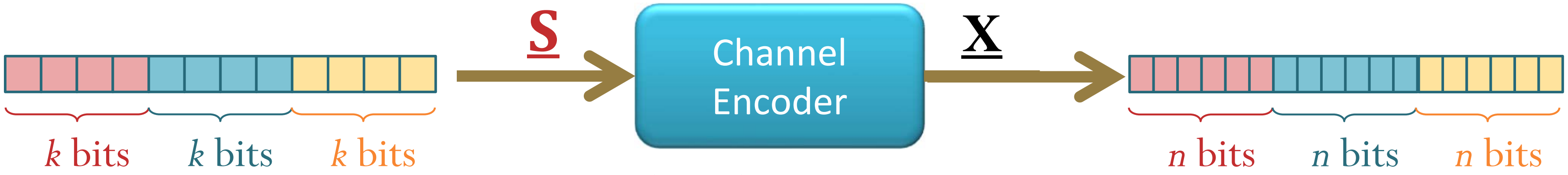
Decoded value

$\underline{\hat{\mathbf{S}}}$

Channel Decoder (Detector)

Receiver

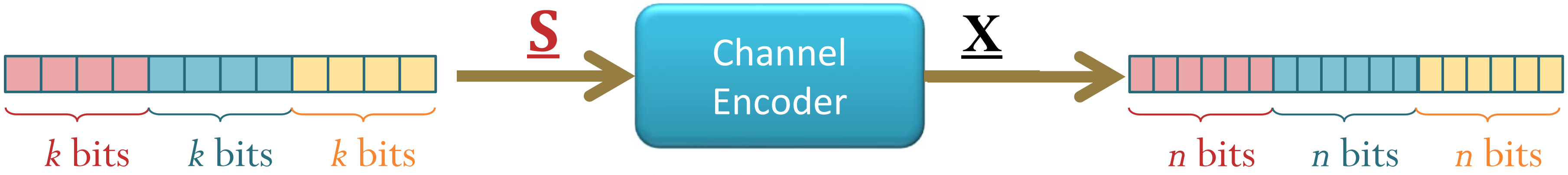
[3.61] Block Encoding



$(\boldsymbol{n}, \boldsymbol{k})$ **code**

$$\textbf{Code rate} = \frac{\boldsymbol{k}}{\boldsymbol{n}}$$

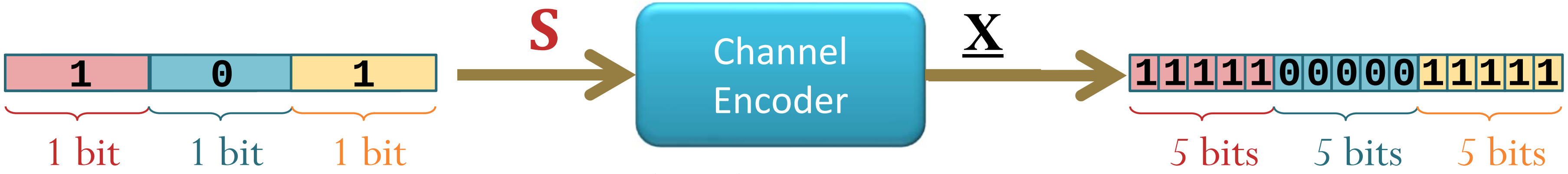
[3.62] Block Encoding



$(\boldsymbol{n}, \boldsymbol{k})$ **code**

Code rate $= \frac{\boldsymbol{k}}{\boldsymbol{n}}$

Example: Repetition Code



$(\mathbf{5}, \mathbf{1})$ **code**

Code rate $= \frac{\mathbf{1}}{\mathbf{5}}$

Vector Notation

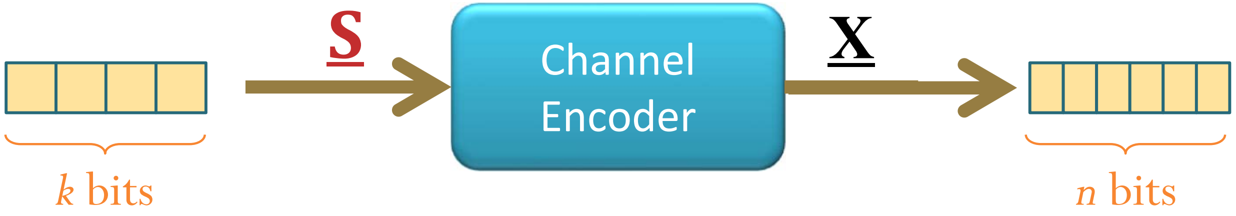
- $\vec{\mathbf{v}}$: column vector $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_i \\ \vdots \\ v_n \end{pmatrix}$
- $\underline{\mathbf{r}}$: row vector $(r_1, r_2, \dots, r_i, \dots r_n)$

$\vec{\mathbf{0}}, \underline{\mathbf{0}}$: the zero vector (the all-zero vector)

$\vec{\mathbf{1}}, \underline{\mathbf{1}}$: the one vector (the all-one vector)

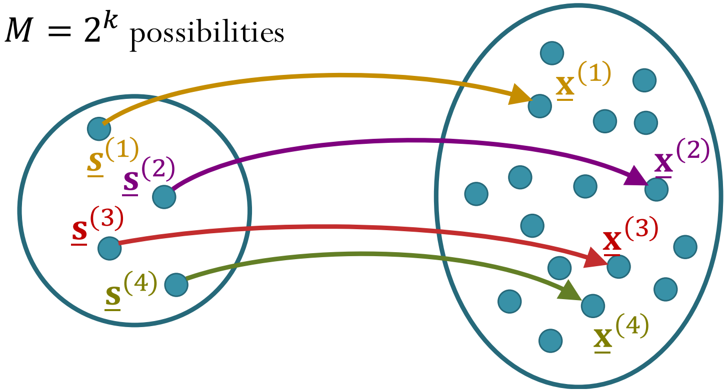
- Subscripts represent element indices inside individual vectors.
  - v_i and r_i refer to the i^{th} elements inside the vectors $\vec{\mathbf{v}}$ and $\underline{\mathbf{r}}$, respectively.
- When we have a list of vectors, we use superscripts in parentheses as indices of vectors.
  - $\vec{\mathbf{v}}^{(1)}, \vec{\mathbf{v}}^{(2)}, \dots, \vec{\mathbf{v}}^{(M)}$ is a list of M column vectors
  - $\underline{\mathbf{r}}^{(1)}, \underline{\mathbf{r}}^{(2)}, \dots, \underline{\mathbf{r}}^{(M)}$ is a list of M row vectors
  - $\vec{\mathbf{v}}^{(i)}$ and $\underline{\mathbf{r}}^{(i)}$ refer to the i^{th} vectors in the corresponding lists.

[3.61] Block Encoding



[Figure 13]

$M = 2^k$ possibilities



Choose $M = 2^k$ from 2^n possibilities to be used as codewords.

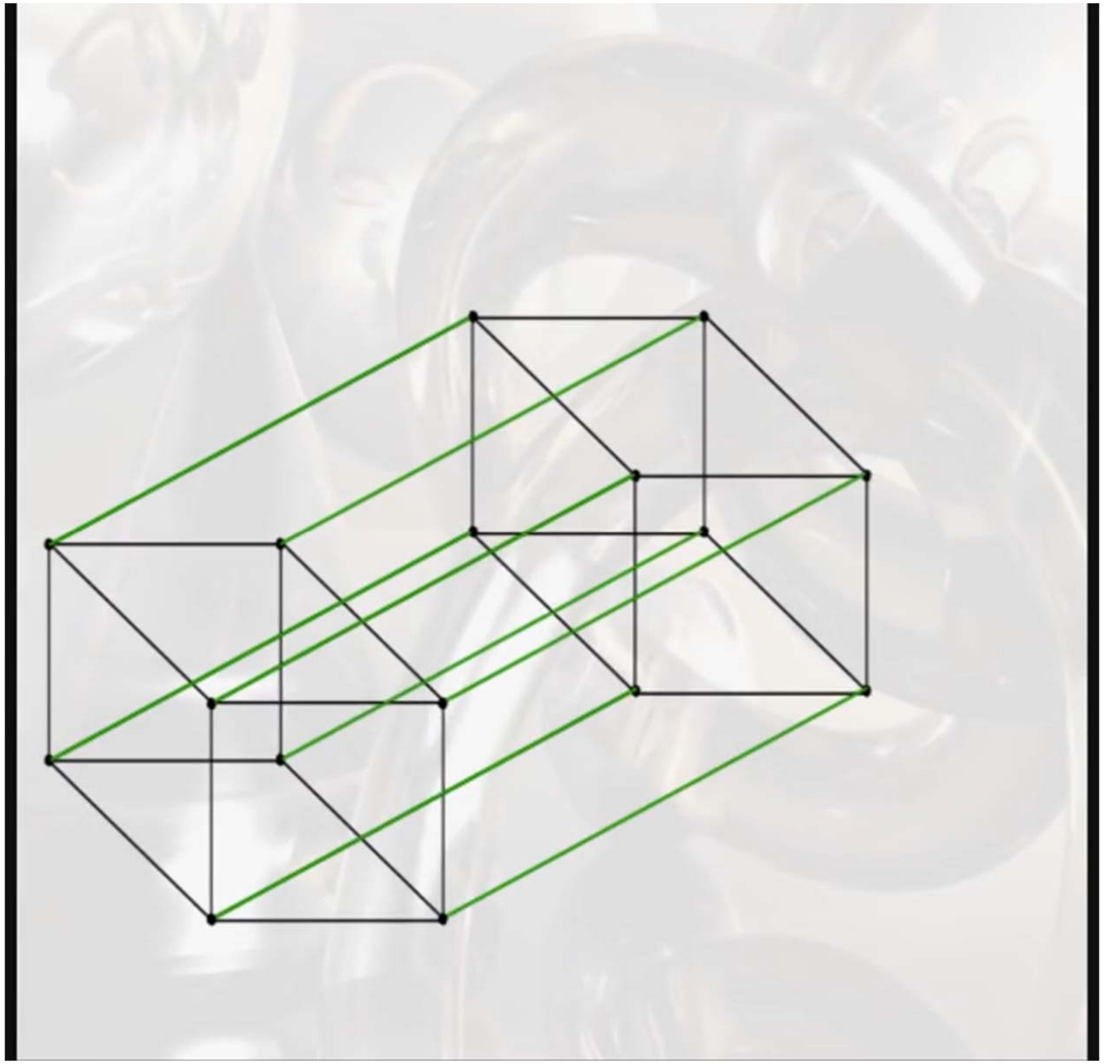
Codebook

index i	info-block $\underline{\mathbf{s}}$	codeword $\underline{\mathbf{x}}$
1	$\underline{\mathbf{s}}^{(1)} = 000\ldots0$	$\underline{\mathbf{x}}^{(1)} =$
2	$\underline{\mathbf{s}}^{(2)} = 000\ldots1$	$\underline{\mathbf{x}}^{(2)} =$
$\vdots$	$\vdots$	$\vdots$
M	$\underline{\mathbf{s}}^{(M)} = 111\ldots1$	$\underline{\mathbf{x}}^{(M)} =$

Hypercube

https://www.youtube.com/watch?v=Q_B5GpsbSQw

Hypercube

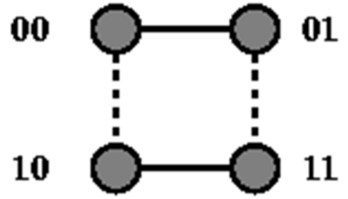


https://www.youtube.com/watch?v=Q_B5GpsbSQw

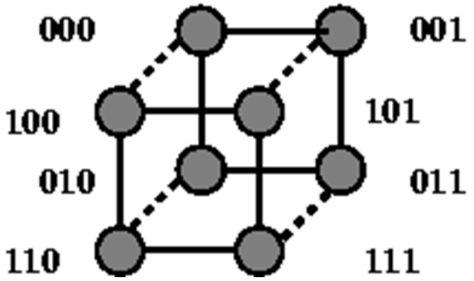
n-bit space

$n = 1$

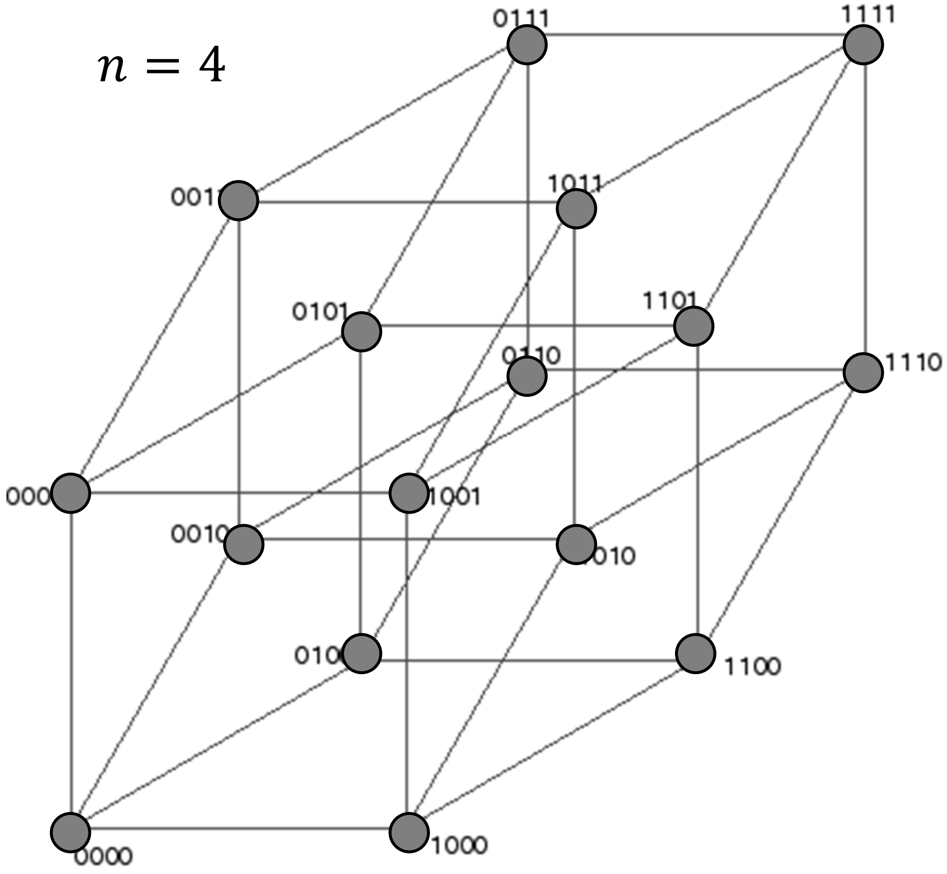
$n = 2$



$n = 3$



$n = 4$



Repetition Code

Example 3.63. Find the codebook and code rate for the encoder which uses repetition code with $n = 5$.

code rate $\to \frac{k}{n} = \frac{1}{5}$

codebook:

codeword

info-block

info-bit

s	$\underline{x}$
0	$\underline{0} \equiv 00000$
1	$\underline{1} \equiv 11111$

$\underline{x}^{(1)}$

$\underline{x}^{(2)}$

$\#\text{rows} = 2^k = 2^1 = 2$

Review

[From ECS315]

One method of reducing the error rate is to use error-correcting codes:

$\underline{Y} = 11000 \Rightarrow \hat{M} = 0$

$\underline{Y} = 01110 \Rightarrow \hat{M} = 1$

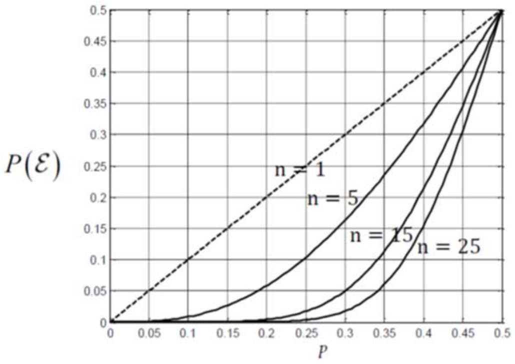
A simple error-correcting code is the ***repetition code***.

$n = 5$

Two ways to calculate the probability of error:

(a) (transmission) error occurs if and only if the number of bits in error are ≥ 3.

$$P(\mathcal{E}) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5\cancel{(1-p)^0}$$



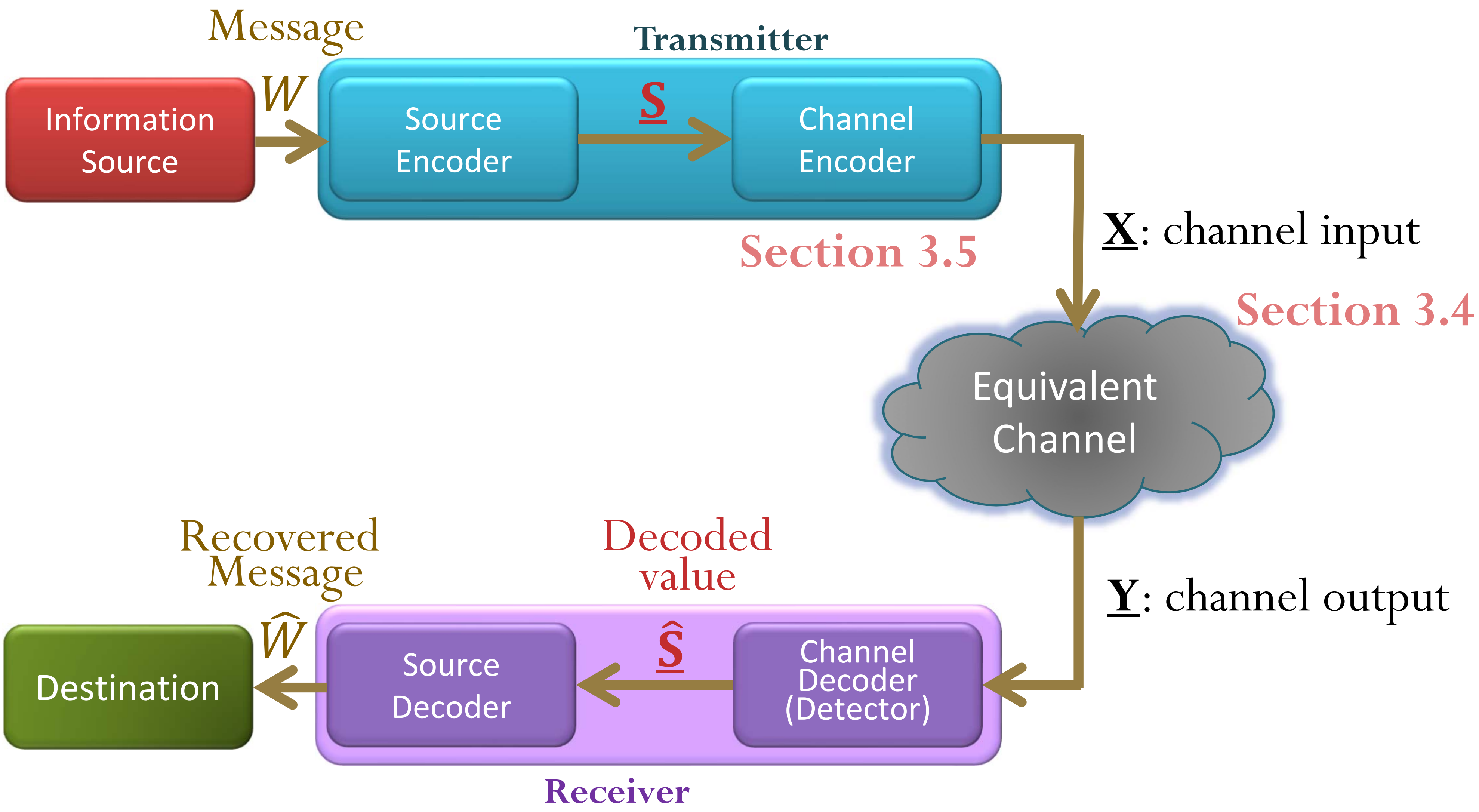
(b) (transmission) error occurs if and only if the number of bits *not* in error are ≤ 2.

$$P(\mathcal{E}) = \binom{5}{0}(1-p)^0p^5 + \binom{5}{1}(1-p)^1p^4 + \binom{5}{2}(1-p)^2p^3$$

with $p = 0.01$

$P(\mathcal{E}) \approx 10^{-5}$

Channel Encoder and Decoder



Message

Transmitter

W

Information Source

Source Encoder

$\underline{S}$

Channel Encoder

Section 3.5

$\underline{\mathbf{X}}$: channel input

Section 3.4

Equivalent Channel

$\underline{\mathbf{Y}}$: channel output

Recovered Message

Decoded value

$\widehat{W}$

$\hat{\underline{S}}$

Destination

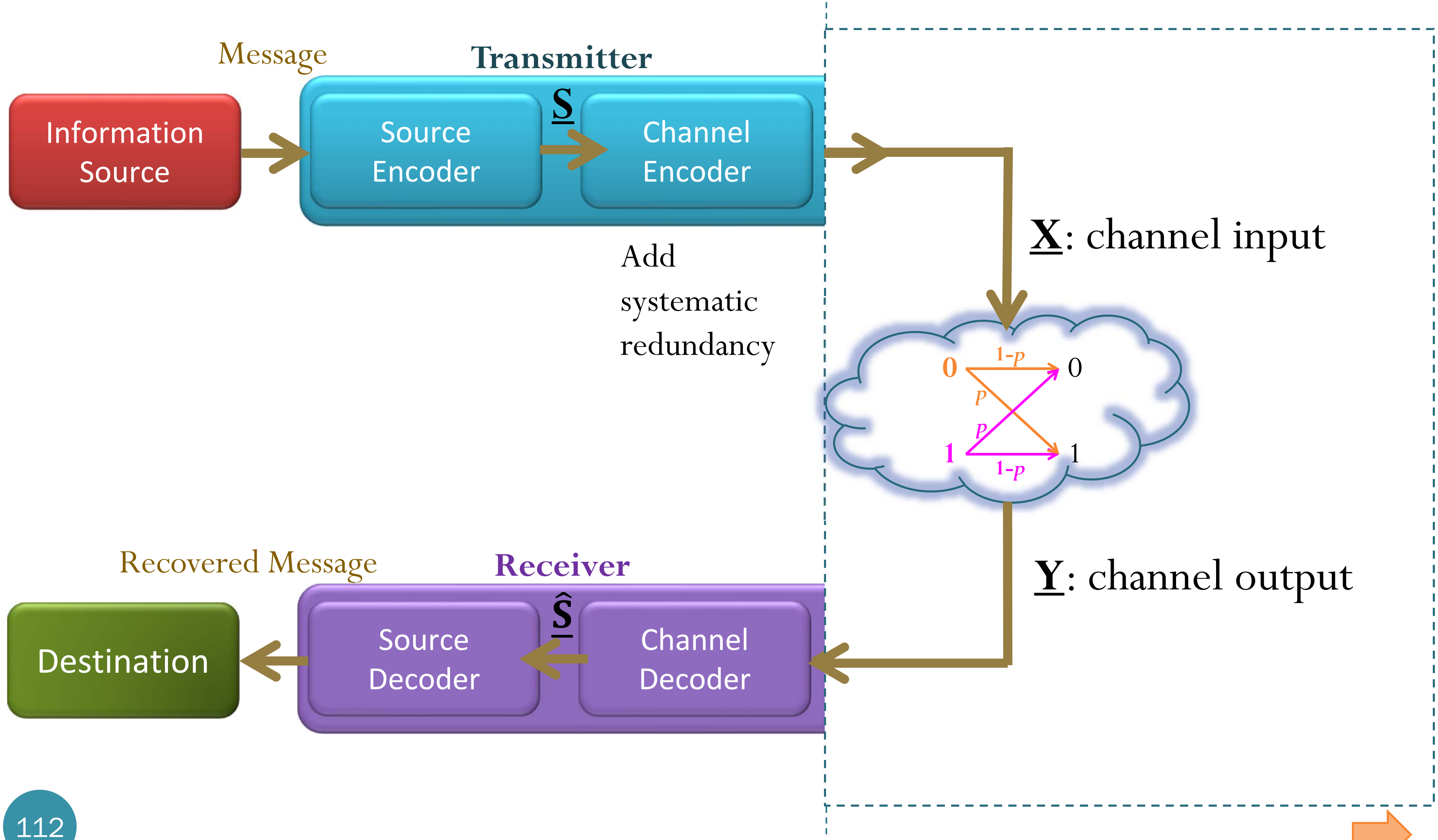
Source Decoder

Channel Decoder (Detector)

Receiver

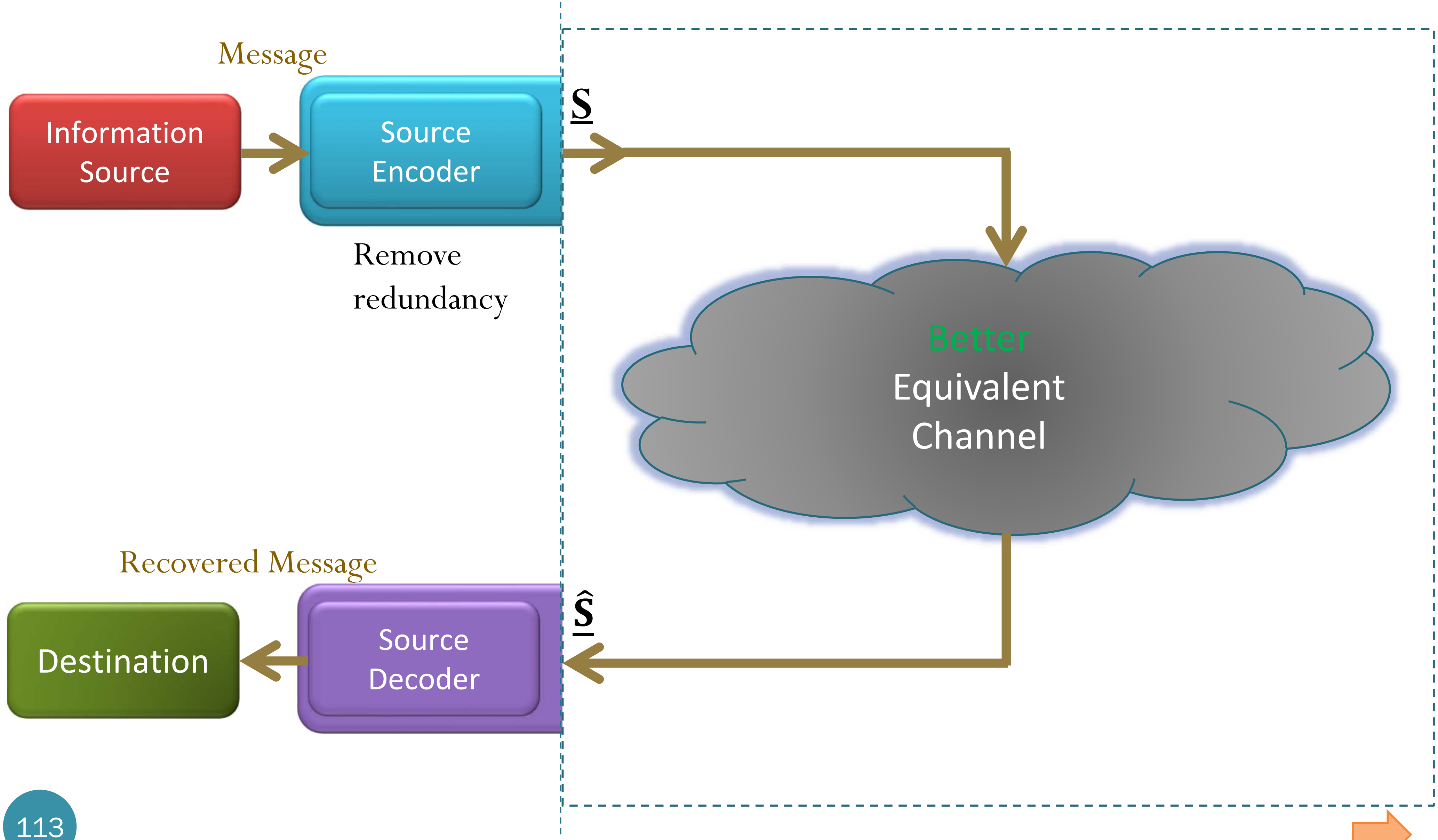
[Figure 14]

Channel Encoder and Decoder

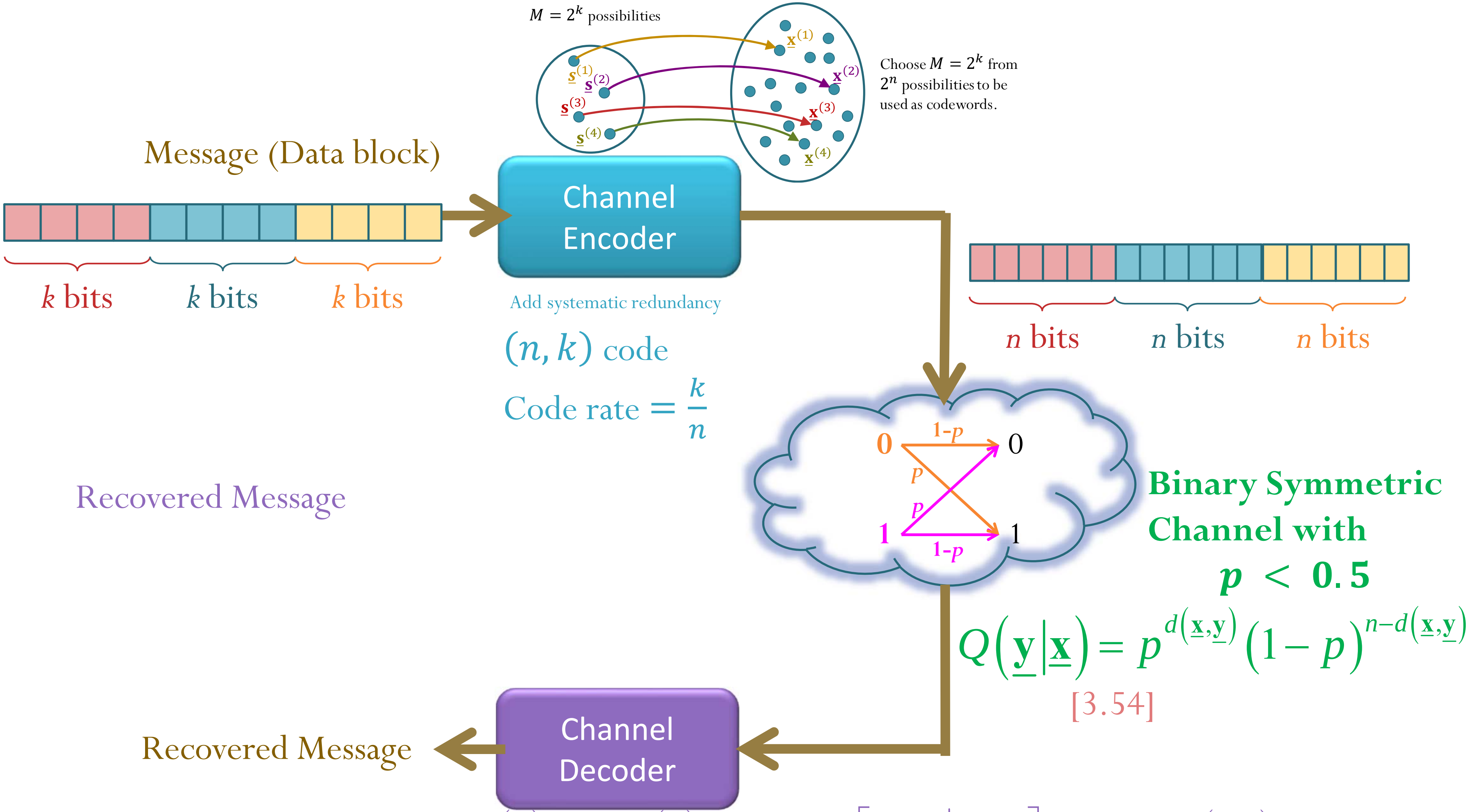


[Figure 14]

Better Equivalent Channel



Review: Channel Encoder and Decoder



$M = 2^k$ possibilities

Choose $M = 2^k$ from 2^n possibilities to be used as codewords.

Message (Data block)

k bits k bits k bits

Channel Encoder

Add systematic redundancy

(n, k) code

$$\text{Code rate} = \frac{k}{n}$$

n bits n bits n bits

Binary Symmetric Channel with

$$p < 0.5$$

$$Q(\underline{\mathbf{y}}|\underline{\mathbf{x}}) = p^{d(\underline{\mathbf{x}},\underline{\mathbf{y}})}(1-p)^{n-d(\underline{\mathbf{x}},\underline{\mathbf{y}})}$$

[3.54]

Recovered Message

Channel Decoder

Recovered Message

$$\hat{\underline{\mathbf{x}}}_{\text{MAP}}(\underline{\mathbf{y}}) = \hat{\underline{\mathbf{x}}}_{\text{optimal}}(\underline{\mathbf{y}}) = \arg\max_{\underline{\mathbf{x}}} P\left[\underline{\mathbf{X}} = \underline{\mathbf{x}} \middle| \underline{\mathbf{Y}} = \underline{\mathbf{y}}\right] = \arg\max_{\underline{\mathbf{x}}} Q(\underline{\mathbf{y}}|\underline{\mathbf{x}})\, p(\underline{\mathbf{x}})$$

$$\hat{\underline{\mathbf{x}}}_{\text{ML}}(\underline{\mathbf{y}}) = \arg\max_{\underline{\mathbf{x}}} P\left[\underline{\mathbf{Y}} = \underline{\mathbf{y}} \middle| \underline{\mathbf{X}} = \underline{\mathbf{x}}\right] = \arg\max_{\underline{\mathbf{x}}} Q(\underline{\mathbf{y}}|\underline{\mathbf{x}})$$

[3.56]

Channel Decoder

[3.57]

$$\hat{\underline{\mathbf{x}}}_{\text{MAP}}\left(\underline{\mathbf{y}}\right) = \hat{\underline{\mathbf{x}}}_{\text{optimal}}\left(\underline{\mathbf{y}}\right)$$
$$= \arg\max_{\underline{\mathbf{x}}} P\left[\underline{\mathbf{X}} = \underline{\mathbf{x}} \middle| \underline{\mathbf{Y}} = \underline{\mathbf{y}}\right]$$
$$= \arg\max_{\underline{\mathbf{x}}} Q\left(\underline{\mathbf{y}} \middle| \underline{\mathbf{x}}\right) p\left(\underline{\mathbf{x}}\right)$$

[Ex 3.54]

BSC

$$Q\left(\underline{\mathbf{y}} \middle| \underline{\mathbf{x}}\right) = p^{d\left(\underline{\mathbf{x}},\underline{\mathbf{y}}\right)}\left(1-p\right)^{n-d\left(\underline{\mathbf{x}},\underline{\mathbf{y}}\right)}$$

ML decoder is the same as the MAP decoder when the codewords are equally likely.

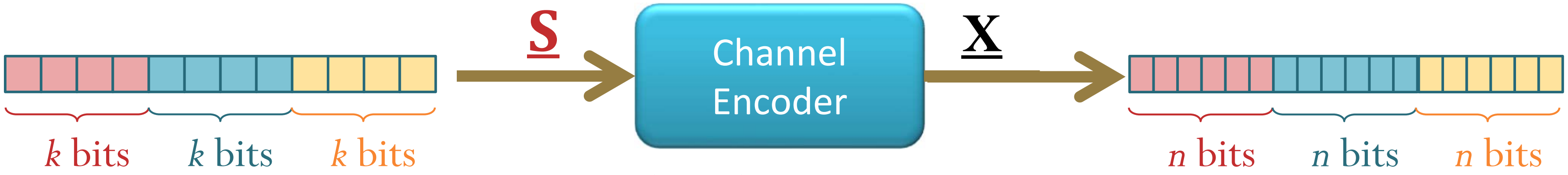
$$\hat{\underline{\mathbf{x}}}_{\text{ML}}\left(\underline{\mathbf{y}}\right) = \arg\max_{\underline{\mathbf{x}}} P\left[\underline{\mathbf{Y}} = \underline{\mathbf{y}} \middle| \underline{\mathbf{X}} = \underline{\mathbf{x}}\right]$$
$$= \arg\max_{\underline{\mathbf{x}}} Q\left(\underline{\mathbf{y}} \middle| \underline{\mathbf{x}}\right)$$

$$\hat{\underline{\mathbf{x}}}_{\min d}\left(\underline{\mathbf{y}}\right) = \arg\min_{\underline{\mathbf{x}}} d\left(\underline{\mathbf{x}},\underline{\mathbf{y}}\right)$$

[Ex 3.56]

For BSC with $p < 0.5$, ML decoder is the same as the min distance decoder

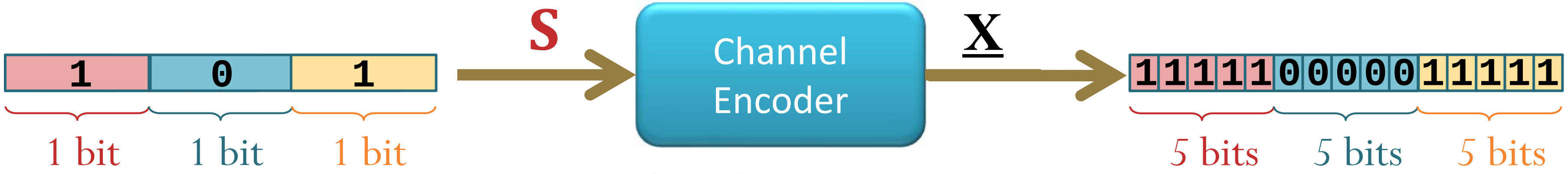
[3.62] Block Encoding



(n, k) code

$$\text{Code rate} = \frac{k}{n}\ \text{[bpcu]}$$

Example: Repetition Code



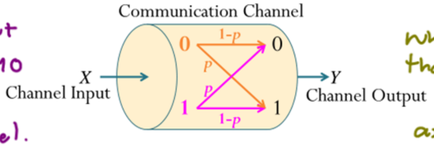
$(5, 1)$ code

$$\text{Code rate} = \frac{1}{5}\ \text{[bpcu]}$$

EES315 2020

Example 6.58. ***Digital communication over unreliable channels***: Consider a digital communication system through the **binary symmetric channel** (BSC) discussed in Example 6.18. We repeat its compact description here.

Suppose you put
010110
into this
channel.



What is the probability
that you get 011010
as output?

010110
↓↓↓↓↓↓
011010

$$(1-p)(1-p)pp(1-p)(1-p) = (1-p)^4p^2$$

Again this channel can be described as a channel that introduces random bit errors with probability p. This p is called the crossover probability.

A crude digital communication system would put binary information into the channel directly; the receiver then takes whatever value that shows up at the channel output as what the sender transmitted. Such communication system would directly suffer bit error probability of p.

In situation where this error rate is not acceptable, error control techniques are introduced to reduce the error rate in the delivered information.

One method of reducing the error rate is to use error-correcting codes:

A simple error-correcting code is the ***repetition code***. Example of such code is described below:

- At the transmitter, the "encoder" box performs the following task:
  - To send a 1, it will send 11111 through the channel.
  - To send a 0, it will send 00000 through the channel.
- When the five bits pass through the channel, it may be corrupted. Assume that the channel is binary symmetric and that it acts on each of the bit independently.
- At the receiver, we (or more specifically, the decoder box) get 5 bits, but some of the bits may be changed by the channel.

  To determine what was sent from the transmitter, the receiver apply the ***majority rule***: Among the 5 received bits,
  - if #1 > #0, then it claims that "1" was transmitted,
  - if #0 > #1, then it claims that "0" was transmitted.

$\underline{Y} = 01101 \Rightarrow \hat{M} = 1$

$\underline{Y} = 00011 \Rightarrow \hat{M} = 0$

$P(\hat{\mathcal{E}})$

Two ways to calculate the probability of error:

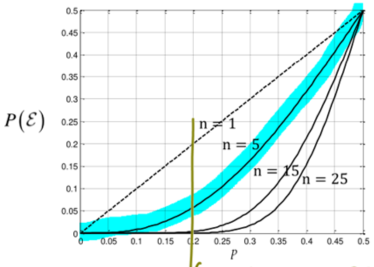
(a) (transmission) error occurs if and only if the number of bits in error from BSC ~~are~~ is ≥ 3.

$$P(\mathcal{E}) = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5\cancel{(1-p)^0}$$

(b) (transmission) error occurs if and only if the number of bits *not* in error ~~are~~ is ≤ 2.

$$P(\mathcal{E}) = \binom{5}{0}(1-p)^0p^5 + \binom{5}{1}(1-p)^1p^4 + \binom{5}{2}(1-p)^2p^3$$

If $p = 0.01$,
$P(\mathcal{E}) \approx 10^{-5}$.



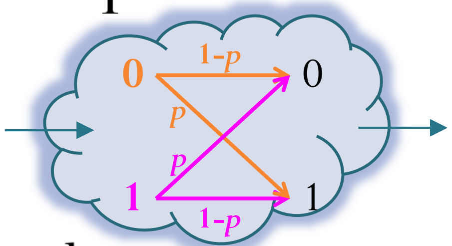
on the same BSC (same p), increasing n provides better (smaller) $P(\mathcal{E})$

Figure 18: Overall bit error probability for a simple system that uses repetition code at the transmitter (repeat each bit n times) and majority vote at the receiver. The channel is assumed to be binary symmetric with bit error probability p.

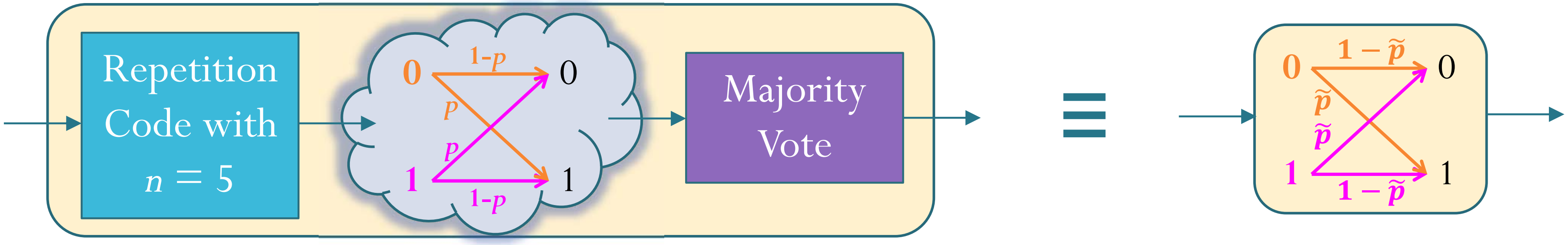
[Figure 14]

Repetition Code

- Original Equivalent Channel:



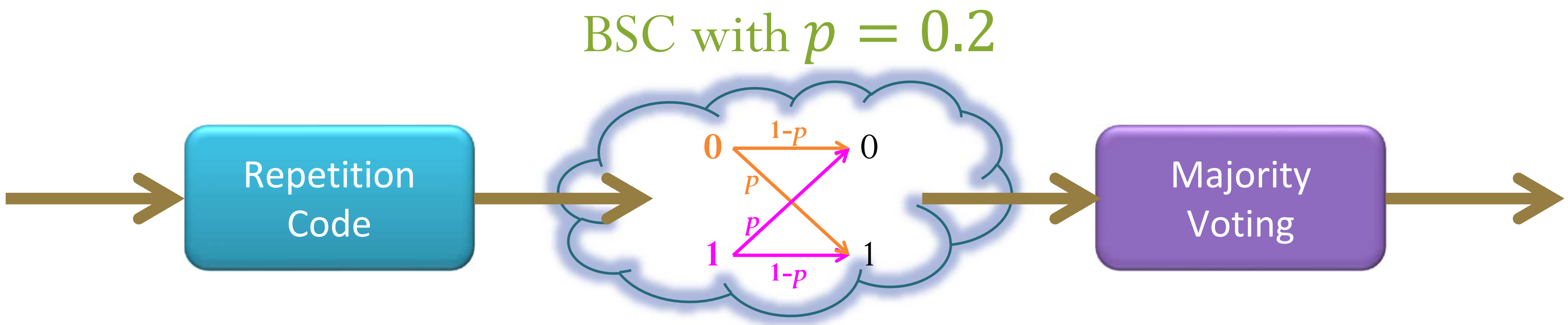
  - BSC with crossover probability $p = 0.01$
- New (and Better) Equivalent Channel:


Repetition Code with $n = 5$

Majority Vote

  - Use repetition code with $n = 5$ at the transmitter
  - Use majority vote at the receiver
  - New BSC with $\tilde{p} = \binom{5}{3}p^3(1-p)^2 + \binom{5}{4}p^4(1-p)^1 + \binom{5}{5}p^5(1-p)^0 \approx 10^{-5}$

Example: Repetition Code



BSC with $p = 0.2$

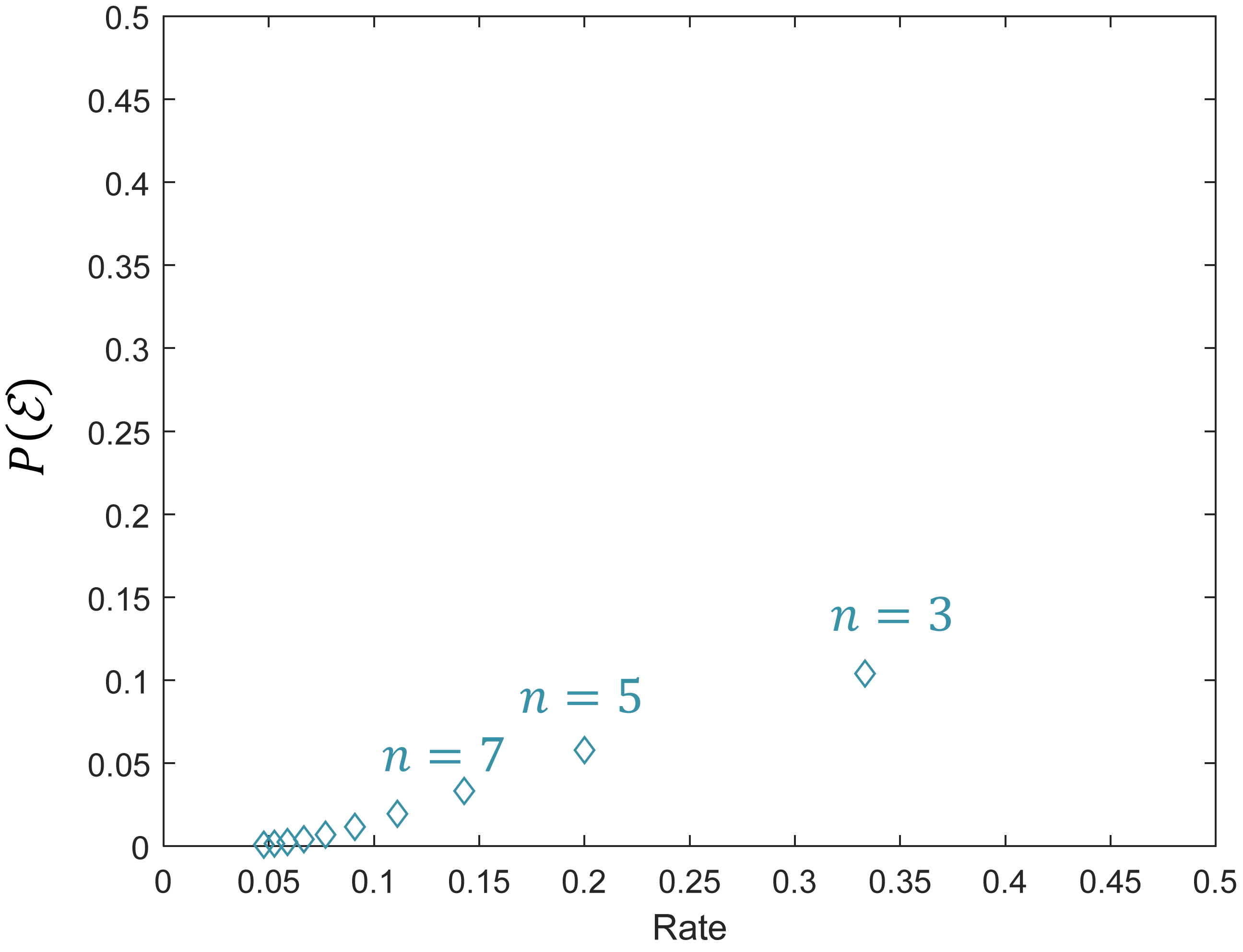
Repetition Code

Majority Voting

n	$P(\mathcal{E})$ Probability that more than half of the bits are in error	Code Rate
1	$p = 0.2$	$\frac{1}{1} = 1$
3	$\binom{3}{2} p^2(1-p) + \binom{3}{3} p^3 \approx 0.1040$	$\frac{1}{3} \approx 0.33$
5	$\binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p)^1 + \binom{5}{5} p^5 \approx 0.0579$	$\frac{1}{5} = 0.2$
7	≈ 0.0333	$\frac{1}{7} \approx 0.1429$
9	≈ 0.0196	$\frac{1}{9} \approx 0.1111$
11	≈ 0.0117	$\frac{1}{11} \approx 0.0909$

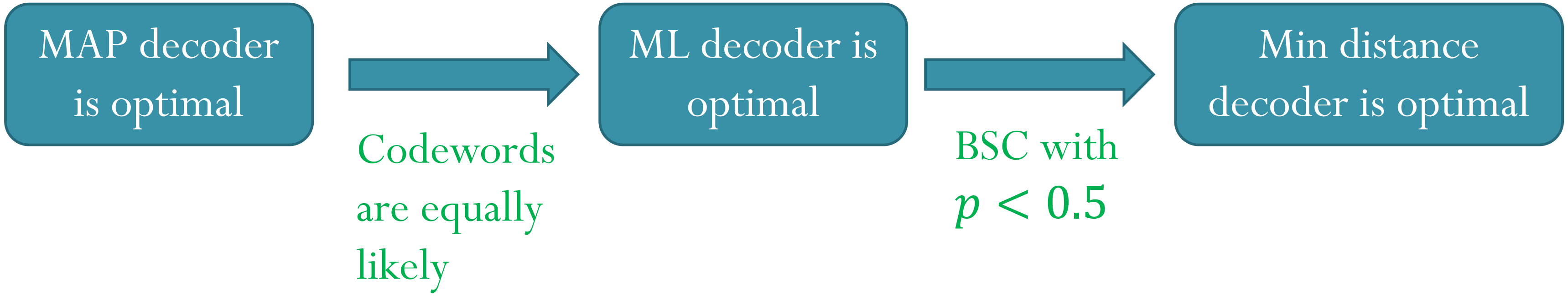
Achievable Performance

BSC with $p = 0.2$ Repetition Code $(k = 1)$



MATLAB: $P(\mathcal{E})$ calc. min dist. dec.

- MATLAB scripts are provided.
- Find $P(\mathcal{E})$ of any binary block code.
- Assumptions:
  - BSC
    - with $p < 0.5$.
  - Codewords are equally likely.
- The (optimal) decoder used is the minimum distance decoder.


MAP decoder is optimal

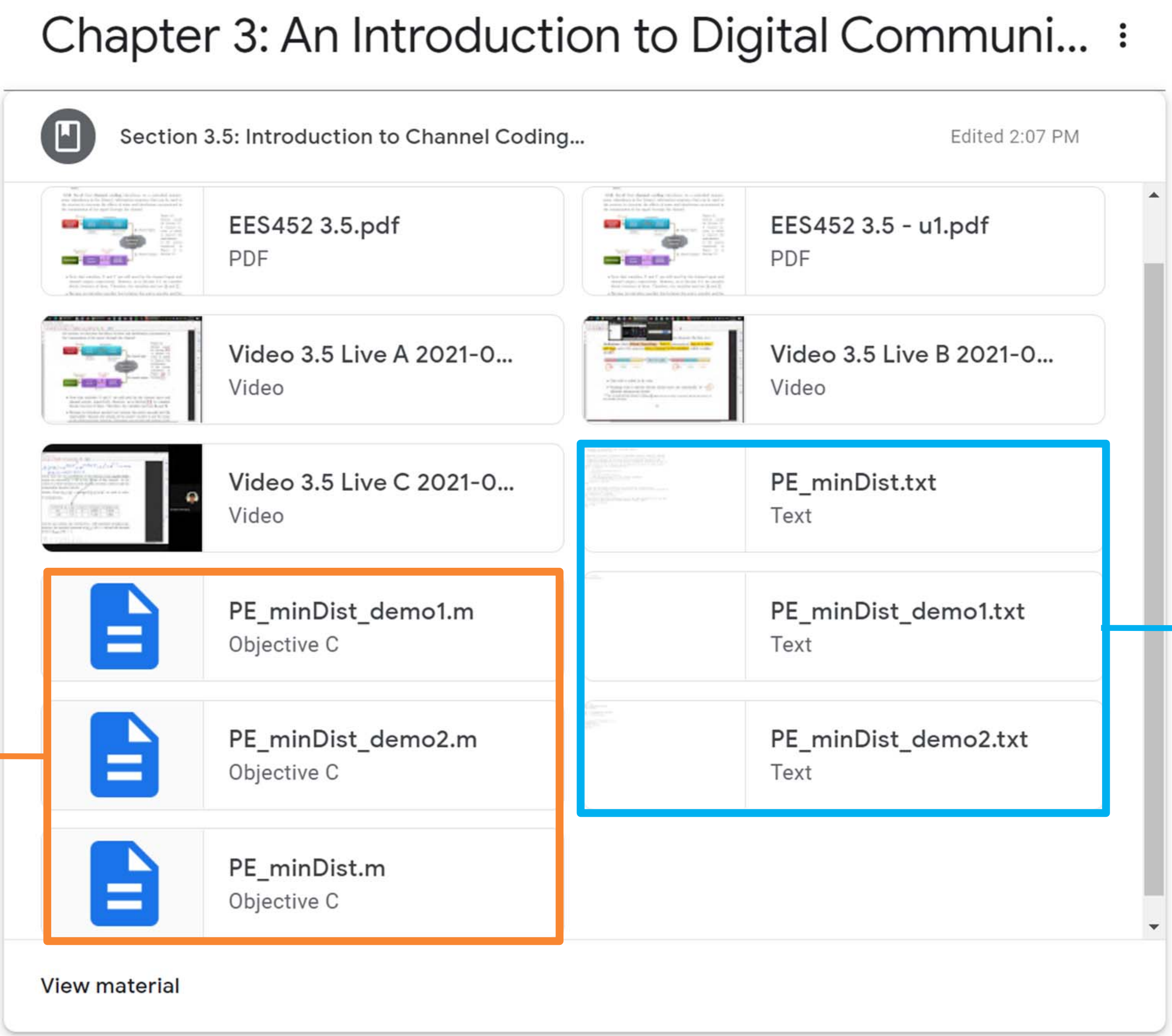
Codewords are equally likely

ML decoder is optimal

BSC with $p < 0.5$

Min distance decoder is optimal

MATLAB



Chapter 3: An Introduction to Digital Communi...

Section 3.5: Introduction to Channel Coding...

Edited 2:07 PM

EES452 3.5.pdf
PDF

EES452 3.5 - u1.pdf
PDF

Video 3.5 Live A 2021-0...
Video

Video 3.5 Live B 2021-0...
Video

Video 3.5 Live C 2021-0...
Video

PE_minDist.txt
Text

PE_minDist_demo1.m
Objective C

PE_minDist_demo1.txt
Text

PE_minDist_demo2.m
Objective C

PE_minDist_demo2.txt
Text

PE_minDist.m
Objective C

View material

Text file (easier to copy on some browsers)

MATLAB

```
close all; clear all;

% EES315 2020 Example 6.58
% EES452 2020 Examples 3.62, 3.67
C = [0 0 0 0 0; 1 1 1 1 1]; % repetition code

p = (1/100);
PE_minDist(C,p)
```

Code C is defined by putting all its (valid) codewords as its rows. For repetition code, there are two codewords: 00..0 and 11..1.

Crossover probability of the binary symmetric channel.

```
>> PE_minDist_demo1

ans =
   9.8506e-06
```

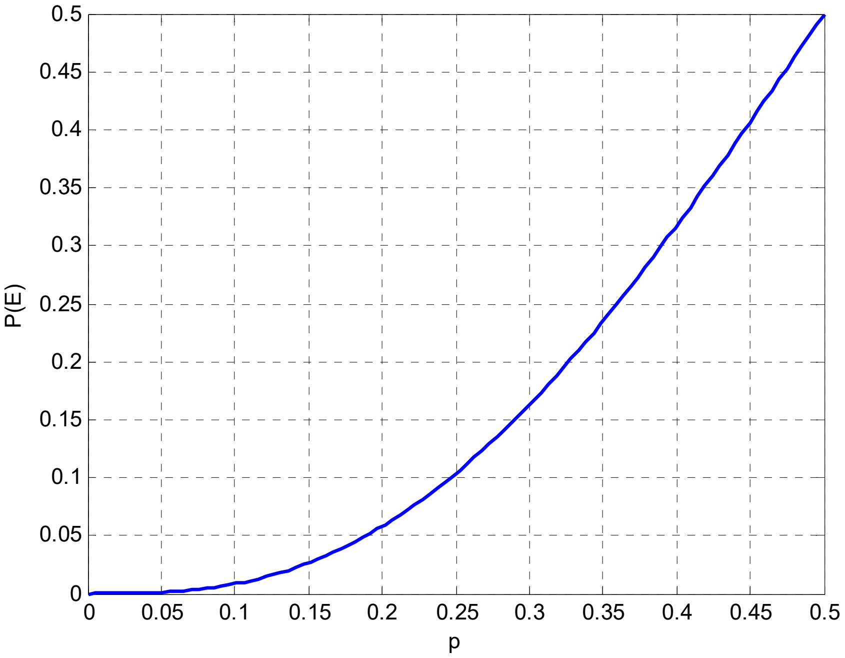
MATLAB

PE_minDist.m

```matlab
function PE = PE_minDist(C,p)
% Function PE_minDist computes the error probability P(E) when code C
% is used for transmission over BSC with crossover probability p.
% Code C is defined by putting all its (valid) codewords as its rows.
M = size(C,1); % the number of (valid) codewords
k = log2(M);
n = size(C,2);

% Generate all possible n-bit received vectors
Y = dec2bin(0:2^n-1)-'0';

% Normally, we need to construct an extended Q matrix. However, because
% each conditional probability in there is a decreasing function of the
% (Hamming) distance, we can work with the distances instead of the
% conditional probability. In particular, instead of selecting the max in
% each column of the Q matrix, we consider min distance in each column.
dminy = zeros(1,2^n); % preallocation
for j = 1:(2^n)
    % for each received vector y,
    y = Y(j,:);
    % find the minimum distance
    % (the distance from y to the closest codeword)
    d = sum(mod(bsxfun(@plus,y,C),2),2);
    dminy(j) = min(d);
end

% From the distances, calculate the conditional probabilities.
% Note that we compute only the values that are to be selected (instead of
% calculating the whole Q first).
n1 = dminy; n0 = n-dminy;
Qmax = (p.^n1).*((1-p).^n0);
% Scale the conditional probabilities by the input probabilities and add
% the values. Note that we assume equally likely input.
PC = sum((1/M)*Qmax);
PE = 1-PC;
end
```

MATLAB

```
close all; clear all;

% ECS315 Example 6.58
% ECS452 Example 3.65
C = [0 0 0 0 0; 1 1 1 1 1];

syms p;
PE = PE_minDist(C,p)
pp = linspace(0,0.5,100);
PE = subs(PE,p,pp);
plot(pp,PE,'LineWidth',1.5)
xlabel('p')
ylabel('P(E)')
grid on
```



```
>> PE_minDist_demo2

PE =
(p - 1)^5 + 10*p^2*(p - 1)^3 - 5*p*(p - 1)^4 + 1
```

Example 3.65

$d\left(\underline{\mathbf{x}}, \underline{\boldsymbol{y}}\right)$

[Ex 3.66]

$$\begin{array}{c|cccccccc} \underline{\mathbf{x}}\backslash\underline{\boldsymbol{y}} & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \hline 011 & 2 & \boxed{1} & \boxed{1} & \boxed{0} & 3 & 2 & 2 & \boxed{1} \\ 100 & \boxed{1} & 2 & 2 & 3 & \boxed{0} & \boxed{1} & \boxed{1} & 2 \end{array}$$

$$Q\left(\underline{\boldsymbol{y}} \middle| \underline{\boldsymbol{x}}\right) = p^{d\left(\underline{\mathbf{x}}, \underline{\boldsymbol{y}}\right)}(1-p)^{n-d\left(\underline{\mathbf{x}}, \underline{\boldsymbol{y}}\right)}$$

[Ex 3.66] [Ex 3.66]

$$\mathbf{Q} = \begin{array}{c|cccccccc} \underline{\mathbf{x}}\backslash\underline{\boldsymbol{y}} & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \hline 011 & 0.032 & \boxed{0.128} & \boxed{0.128} & \boxed{0.512} & 0.008 & 0.032 & 0.032 & \boxed{0.128} \\ 100 & \boxed{0.128} & 0.032 & 0.032 & 0.008 & \boxed{0.512} & \boxed{0.128} & \boxed{0.128} & 0.032 \end{array} \begin{array}{c} \xrightarrow{\times 0.5} \\ \xrightarrow[\times 0.5]{} \end{array} \begin{array}{c|cccccccc} \underline{\mathbf{x}}\backslash\underline{\boldsymbol{y}} & 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ \hline 011 & 0.016 & \boxed{0.064} & \boxed{0.064} & \boxed{0.256} & 0.004 & 0.016 & 0.016 & \boxed{0.064} \\ 100 & \boxed{0.064} & 0.016 & 0.016 & 0.004 & \boxed{0.256} & \boxed{0.064} & \boxed{0.064} & 0.016 \end{array} = \mathbf{P}$$

```
close all; clear all;

C = [0 1 1; 1 0 0];

p = 0.2;
PE_minDist(C,p)
```

```
>> PE_minDist_demo3

ans =
    0.1040
```

$P(\mathcal{C}) = 0.8960$

$P(\mathcal{E}) = 0.1040$

Example 3.66d

```
close all; clear all;

% ECS452 Example 3.66d
C = [0 0 0 0 0; 0 1 0 0 0; 1 0 0 0 1; 1 1 1 1 1];

p = (0.1);
PE_minDist(C,p)
```

```
>> PE_minDist_demo5_1

ans =

    0.1252
```

$P(\mathcal{E}) = 0.1252$

```
close all; clear all;

% ECS452 Example 3.66d
C = [0 0 0 0 0; 0 1 0 0 0; 1 0 0 0 1; 1 1 1 1 1];

syms p;
PE = expand(PE_minDist(C,p))
```

```
>> PE_minDist_demo5_2

PE =

2*p^4 - 5*p^3 + 3*p^2 + p
```

$P(\mathcal{E}) = 2p^4 - 5p^3 + 3p^2 + p$

Is there a better code for $k = 2, n = 5$?

Searching for the best encoder

- Now that we have MATLAB function **PE_minDist**,
  for specific values of n, k,
  we can try to search for the encoder that minimizes the error probability.
- Recall that, from **Example 3.63**, there are
  $$\binom{2^n}{M} = \binom{2^n}{2^k} = 35{,}960$$ “reasonable” encoders.
- Even for small n and k, this is a large space to look at every possible cases.

s	**x**
00	? ? ? ? ?
01	? ? ? ? ?
10	? ? ? ? ?
11	? ? ? ? ?

2^k rows

n columns

Each “?” can be 0 or 1.